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A NOTE ON THE BALANCEDNESS AND THE CONCAVITY OF HIGHWAY GAMES

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A Note on the Balancedness and the Concavity of Highway Games*

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Abstract

A *highway problem* is determined by a connected graph which provides all potential entry and exit vertices and all possible edges that can be constructed between vertices, a cost function on the edges of the graph and a set of players, each in need of constructing a connection between a specific entry and exit vertex. Mosquera and Zarzuelo (2006) introduce *highway problems* and the corresponding cooperative cost games called *highway games* to address the problem of fair allocation of the construction costs in case the underlying graph is a chain. In this note, we study the concavity and the balancedness of highway games on more general graphs. A graph G is called highway-game concave if for each highway problem in which G is the underlying graph the corresponding highway game is concave. The main result of our study is that a graph is highway-game concave if and only if it is weakly triangular. Moreover, we provide sufficient conditions on highway problems defined on cyclic graphs such that the corresponding highway games are balanced.

Keywords: cooperative games, highway games, cost sharing.
JEL code: C71.

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1 Introduction

Mosquera and Zarzuelo (2006) address the problem of fair allocation of the construction costs of a highway network. For this aim, they formally consider highway problems and analyze the corresponding cooperative cost games called highway games. In a highway problem, the possibilities regarding the construction of the highway network are determined by a connected graph. The set of vertices of the graph represents the potential entry and exit points and the edges in the graph represent the possible highway connections that can be constructed. Each edge in the graph has an associated cost which in general will depend on its length or the geographical properties that may affect the construction costs of the highway. Each player in a highway problem has to establish a connection between two vertices in the graph, i.e., between his entry and exit point. Given a highway problem, a corresponding highway game is defined as a cooperative cost game which associates to each coalition of players the total cost of the cheapest selection of edges in the graph which connects the entry and exit point of every member of the coalition. Mosquera and Zarzuelo (2006) restricted attention to highway problems in which the underlying graph is a chain. In this setting, there is only one path between an entry and exit point.

In this note, we study highway problems which allow for more general graphs. In particular, these graphs may contain cycles and hence, there will exist multiple paths between some entry and exit points. Note that, in this setting, a coalition of players can further reduce the joint construction costs by an optimal coordination of paths to construct. That is, the joint minimal cost of a coalition is now obtained as a result of solving a combinatorial optimization problem. Concavity (and hence balancedness) is a straightforward result for highway games induced by chain graphs. However, balancedness and concavity results are not immediate even for the simplest generalization of highway problems on chain graphs to cyclic graphs.

We start our analysis of highway games by investigating their concavity properties. A cooperative cost game is called concave if it exhibits the property that the incentives to join a coalition increases as the coalition becomes larger. We proceed as Herer and Penn (1995) on traveling salesman problems and Granot et al. (1999) on Chinese postman problems, and focus on the question for which class of graphs the corresponding games are always concave. We define a graph to be highway game-concave (HG-concave), if for every player set, for every choice of entry and exit points for the players and for every cost specification, the corresponding highway game is concave. The main result of this note is that a graph is HG-concave if and only if it is weakly triangular. Here, a graph is called weakly triangular if it is weakly cyclic, i.e., every edge in the graph is contained in at most one cycle and, moreover, if every cycle is a triangle, i.e., every cycle is composed of three edges.

We then investigate the core of the highway games. The core of a cost game is defined as the set of cost allocations that are stable in the sense that no coalition of players can do better by splitting off. A game with a nonempty core is called balanced. Highway games induced by chains and trees, that is the graphs which provide only one path between any two vertices, are always balanced. However, highway games induced by graphs which allow for multiple paths between vertices need not be balanced in general. In this note, we first focus on highway problems defined on cycles and provide several sufficient conditions such that the corresponding highway games are balanced. Finally, we prove that the same conditions are also sufficient for the balancedness of highway games induced by weakly cyclic graphs.

The outline of the paper is as follows. Section 2 recalls basic notions from cooperative game theory and graph theory and formally introduces highway problems and highway games. Section 3 presents the main result of the paper: the characterization of HG-concave graphs. Section 4 presents our results regarding the balancedness of highway games on weakly cyclic graphs. Section 5 concludes.

2 Highway Problems and Highway Games

In this section, we formally define highway problems and the corresponding cooperative cost games, called highway games.

2.1 Preliminaries

A cooperative (cost) game is a pair (N, c), where N is a nonempty, finite set of players and c is a mapping, $c : 2^N \to \mathbb{R}$ with $c(\emptyset) = 0$. A *coalition* is a set of players $S \subset N$ and N is called *the grand coalition*. For any coalition $S \subset N$, $c(S)$ is interpreted as the minimal joint cost of coalition S. A game (N, c) is *monotonic* if $c(S) \geq c(T)$ for every $S, T \in 2^N$ with $T \subset S$ and it is called *subadditive* if $c(S) + c(T) \geq c(S \cup T)$ for every $S, T \in 2^N$ with $T \cap S = \emptyset$. A game (N, c) is *concave* if $c(S \cup \{i\}) - c(S) \leq c(T \cup \{i\}) - c(T)$ for every $i \in N$ and $S, T \subset N \backslash \{i\}$ with $T \subset S$. Equivalently, a game (N, c) is concave if $c(T \cup S) + c(T \cap S) \leq c(T) + c(S)$ for every $S, T \subset N$.

The core $C(c)$ of a game (N, c) is defined as the set of efficient cost allocations for which no coalition has an incentive to split off from the grand coalition, i.e., $C(c) = \{x \in R^N | \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \in 2^N\}$. A game with a nonempty core is called balanced. In particular, concave games are balanced.

An (undirected) graph G is a pair (V, E), where V is a nonempty and finite set of vertices and E is a subset of all edges $\{i, j\}$ with $i, j \in V$, $i \neq j$. Let $G = (V, E)$ be a graph. A *path* in G between vertices i and j is a collection of edges $\{\{i_0, i_1\}, \{i_1, i_2\}, ..., \{i_{k-1}, i_k\}\}$ such that $i_0 = i$, $i_k = j$, $\{i_{q-1}, i_q\} \in E$ for all $q \in \{1, ..., k\}$ and all intermediate vertices are distinct, i.e., $i_q \neq i_r$ for all $q, r \in \{1, 2, ..., k-1\}$ with $q \neq r$. A *cycle* in G is a path from i to i for some $i \in V$. A graph is called *weakly cyclic* if every edge in the graph is contained in at most one cycle. Vertices $i, j \in V$ are said to be *connected* in G if there exists a path between i and j in G. G is *connected* if any two vertices in V are connected. If G is connected, an edge $e \in E$ is called a *bridge* in G if the graph $(V, E \backslash \{e\})$ is not connected.

A *subgraph* of G is a graph $G' = (V', E')$ with $V' \subset V$, $V' \neq \emptyset$ and E' is a subset of all edges $\{i,j\} \in E$ with $i,j \in V'$, $i \neq j$.

2.2 Highway Games

A highway problem is defined as a tuple $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i\in N}, w)$. $N = \{1, ..., n\}$ is a nonempty, finite set of players and $G = (V, E)$ is a connected graph. The graph G determines the possibilities regarding the construction of the highway network. That is, any constructed highway network has to be a subgraph of G. Note that G need not be the complete graph, since the construction of some edges may be infeasible due to geographic or socioeconomic reasons. For each player $i \in N$, s_i and t_i are vertices in G and they are called the connection vertices of i. The connection vertices of player i represent the locations (think of entry and exit) that i has to establish a connection in between. Furthermore, $w : E \to \mathbb{R}_+$ is called a *cost function* and associates to each edge, $e \in E$, the nonnegative cost $w(e)$ of constructing e. The total cost of constructing a set of edges $E' \subset E$ is abbreviated by $w(E') = \sum_{e\in E'} w(e)$.

In a highway problem, a coalition S of cooperating players will construct the cheapest set of edges that connects the connection points of every member of S. Therefore, given a highway problem $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i\in N}, w)$, the corresponding highway game (N, c_Γ) is defined by

$$c_\Gamma(S) = min_{E'\subset E}\{w(E')|s_i \text{ and } t_i \text{ are connected in } (V, E') \text{ for every } i \in S\} \tag{1}$$

for all $S \subset N$. Clearly, (N, c_Γ) is subadditive and monotonic.

Example 2.1 Let $G = (V, E)$ be a cyclic graph with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\}$. The construction costs of the edges are given by: $w(\{v_1, v_2\}) = w(\{v_2, v_3\}) = 2$ and $w(\{v_3, v_4\}) = w(\{v_4, v_1\}) = 3$. Consider $N = \{1, 2, 3\}$ with $s_1 = v_1$, $t_1 = v_3$, $s_2 = v_2$, $t_2 = v_3$, $s_3 = v_4$, $t_3 = v_1$. The corresponding highway problem $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i\in N}, w)$ is depicted in Figure 1.

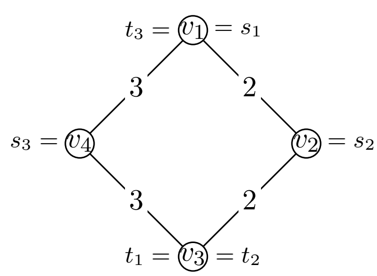

Figure 1: A highway problem with three players

Consider player 1. There are two paths in G between player 1's connection vertices v_1 and v_3: $\{\{v_1, v_2\}, \{v_2, v_3\}\}$ and $\{\{v_1, v_4\}, \{v_4, v_3\}\}$. Since player 1 will not construct any superfluous edges, $c_\Gamma(\{1\})$ is the minimum of $w(\{v_1,v_2\})+w(\{v_2,v_3\})=4$ and $w(\{v_1,v_4\})+w(\{v_4,v_3\})=6$, i.e., $c_\Gamma(\{1\})=4$.

Next, consider the coalition $\{1,3\}$. Clearly, players 1 and 3 will construct the set of edges $\{\{v_1,v_4\},\{v_3,v_4\}\}$. Hence, $c_\Gamma(\{1,3\})=6$.

The complete corresponding highway game (N,c_Γ) is given below:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
c_Γ(S)	4	2	3	4	6	5	7

Observe that this highway game is not concave:

$$c_\Gamma(\{1,2\})+c_\Gamma(\{1,3\})<c_\Gamma(\{1\})+c_\Gamma(\{1,2,3\}).$$

$\diamond$

Recall that when the underlying graph is a chain as it is the case for the highway problems considered by Mosquera and Zarzuelo (2006), the induced highway games are concave. In chain graphs, there exists only one path between the connection vertices of a player. However, if there are multiple paths between the connection vertices of players in the underlying graph, the above example illustrates that players can select different paths in different coalitions and this may lead to the violation of concavity conditions of the induced highway game.

3 HG-Concavity

In this section, we characterize HG-concave graphs. Recall that a graph G is HG-concave if for every highway problem $(N,G,\{\{s_i,t_i\}\}_{i\in N},w)$, the corresponding highway game (N,c_Γ) is concave. Explicitly, we show that a graph is HG-concave if and only if it is weakly triangular, i.e., every edge in the graph is contained in at most one cycle and every cycle has three edges.

For this aim, we first show that every highway game on a triangle is concave.

Lemma 3.1 *Let $\Gamma=(N,G,\{\{s_i,t_i\}\}_{i\in N},w)$ be a highway problem where G is a cyclic graph with three edges. Then, the corresponding highway game (N,c_Γ) is concave.*

Proof. Without loss of generality, assume that $s_i\neq t_i$ for all $i\in N$. It can easily be observed that

(i) for every coalition, it is optimal to construct either one edge in G or the two cheaper edges in G;

(ii) for a coalition $S\subset N$, if it is optimal to construct one edge $\{u,v\}$ in G, then $\{s_i,t_i\}=\{u,v\}$ for every $i\in S$;

(iii) for a coalition $S \subset N$, if it is optimal to construct the two cheaper edges in G, then constructing the two cheaper edges is optimal for any superset of S, too.

Now, we will show that $c_\Gamma(S \cup T) + c_\Gamma(S \cap T) \leq c_\Gamma(S) + c_\Gamma(T)$ for every $S, T \subset N$, i.e., that the corresponding highway game (N, c_Γ) is concave. Take $S, T \subset N$ and assume that $S \cap T \neq \emptyset$. For, if $S \cap T = \emptyset$, then the inequality follows directly from the subadditivity of (N, c_Γ).

Firstly, (i) implies that, for any coalition $K \subset N$, $c(K)$ is either equal to the sum of the costs of the two cheaper edges in G or equal to the cost of one of the three edges in G.

If both $c_\Gamma(S)$ and $c_\Gamma(T)$ are equal to the sum of the costs of the two cheaper edges in G, then the inequality follows from the monotonicity of c_Γ and (iii). If only one of $c_\Gamma(S)$ and $c_\Gamma(T)$ is equal to the sum of the costs of the two cheaper edges in G and the other is equal to the cost of one edge, say e, in G, then (iii) implies that $c_\Gamma(S \cup T)$ is equal to the sum of the costs of the two cheaper edges in G and (ii) implies that $c_\Gamma(S \cap T)$ is also equal to the cost of e. Hence, $c_\Gamma(S \cup T) + c_\Gamma(S \cap T) = c_\Gamma(S) + c_\Gamma(T)$.

Lastly, assume that $c_\Gamma(S)$ and $c_\Gamma(T)$ are equal to the cost of an edge in G. Then, since $S \cap T \neq \emptyset$, (ii) implies that $c_\Gamma(S)$ and $c_\Gamma(T)$ have to be equal to the cost of the same edge in G. Then, both $c_\Gamma(S \cup T)$ and $c_\Gamma(S \cap T)$ are equal to the cost of the same edge, too. Hence, $c_\Gamma(S \cup T) + c_\Gamma(S \cap T) = c_\Gamma(S) + c_\Gamma(T)$. □

We now discuss some properties of weakly cyclic graphs. Let $G = (V, E)$ be a weakly cyclic graph. Clearly, each edge in G is either a bridge edge or belongs to a cycle in G. Let $C(G)$ denote the set of cycles in G and $BE(G)$ denote the set of bridge edges in G. Observe that every path in G which connects two vertices has to pass through (has a common edge with) the same set of cycles and the same set of bridge edges in G. More specifically, every path that connects the same two vertices, passes through the same cycles and the same bridge edges but the edges followed in cycles may differ. Moreover, every path that connects the same two vertices in G enter and leave a cycle that they pass through at the same vertices.

Before presenting the main result of this section, we will show that for every highway problem on a weakly cyclic graph, the corresponding highway game is equal to the sum of specific sub-highway games on each cycle and on each bridge edge in the graph. These sub-highway games are formally defined as follows.

Consider a highway problem $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i \in N}, w)$ where G is a weakly cyclic graph. Let C be a cycle in G. There exists a set of vertices $V_C = \{v_1, v_2, ..., v_k\} \subset V$ ($k \geq 3$) such that $C = \{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_k, v_1\}\}$. Now, the sub-highway problem with respect to C is defined by $\Gamma^C = (N, (V_C, C), \{\{s_i^C, t_i^C\}\}_{i \in N}, w_{|C})$, where $w_{|C}$ is the restriction of the cost function w to the edges in C. For each player $i \in N$, if the paths connecting s_i and t_i pass through C, then s_i^C and t_i^C are the vertices in C at which the paths connecting s_i and t_i enter and leave C. If the paths connecting s_i and t_i do not pass through C, then we set $s_i^C = t_i^C = v_1$.

Next, let $e = \{u, v\}$ be a bridge edge in G. Then, the sub-highway problem with respect to e is defined by $\Gamma^e = (N, (\{u, v\}, \{e\}), \{\{s_i^e, t_i^e\}\}_{i\in N}, w_{|\{e\}})$. Set $s_i^e = u$ and $t_i^e = v$ if the paths connecting s_i and t_i pass through e. Otherwise, set $s_i^e = t_i^e = u$.

Lemma 3.2 *Let $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i\in N}, w)$ be a highway problem where G is a weakly cyclic graph. Then, $c_\Gamma(S) = \sum_{C\in C(G)} c_{\Gamma^C}(S) + \sum_{e\in BE(G)} c_{\Gamma^e}(S)$ for every $S \subset N$.*

We omit the proof of Lemma 3.2 since it is straightforward.

We are now ready to present the main result of this section.

Theorem 3.1 *A graph G is HG-concave if and only if it is weakly triangular.*

Proof. We first show the if-part. Let $G = (V, E)$ be a weakly triangular graph and consider a highway problem $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i\in N}, w)$. We will show that the corresponding highway game (N, c_Γ) is concave.

We know by Lemma 3.2 that $c_\Gamma(S) = \sum_{C\in C(G)} c_{\Gamma^C}(S) + \sum_{e\in BE(G)} c_{\Gamma^e}(S)$ for every $S \subset N$. By Lemma 3.1, we have that c_{Γ^C} is concave for every triangle $C \in C(G)$ and we also know that highway games induced by chains are concave. In particular, c_{Γ^e} is concave for every $e \in BE(G)$. We may conclude that c_Γ is concave, since it is a non-negative linear combination of concave games.

For the only-if part of the proof, choose a graph $G = (V, E)$ that is not weakly triangular. Now, we construct a player set N, connection vertices s_i, t_i for each player i in N and a cost function w such that the highway game corresponding to the highway problem $(N, G, \{\{s_i, t_i\}\}_{i\in N}, w)$ is not concave.

Since G is not weakly triangular, it contains a cycle with more than three edges. Let (V', E') be a subgraph of G corresponding to one such cycle C, i.e., $V' = \{v_1, v_2, ..., v_k\}$ with $k \geq 4$ and $E' = \{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{k-1}, v_k\}, \{v_k, v_1\}\}$. Let the player set be $N = \{1, 2, 3\}$ and let $s_1 = v_1$, $t_1 = v_3$, $s_2 = v_2$, $t_2 = v_3$ and $s_3 = v_k$, $t_3 = v_1$. Define the cost function w by:

$$w(e) = \begin{cases} 2 & \text{if } e \in \{\{v_1, v_2\}, \{v_2, v_3\}\} \\ 0 & \text{if } e \in \{\{v_3, v_4\}, ..., \{v_{k-2}, v_{k-1}\}\} \\ 3 & \text{if } e \in \{\{v_k, v_1\}, \{v_k, v_{k-1}\}\} \\ 100 & \text{if } e \notin C \end{cases}$$

Figure 2 provides a figure depicting a part of the highway problem $(N, G, \{\{s_i, t_i\}\}_{i\in N}, w)$. Now, it can easily be shown that the highway game corresponding to the highway problem $(N, G, \{\{s_i, t_i\}\}_{i\in N}, w)$ is equal to the highway game presented in Example 2.1, which is not concave. $\square$

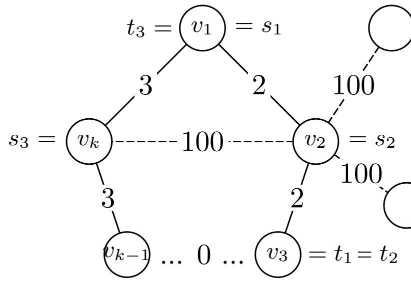

Figure 2: An auxiliary figure for the proof of Theorem 3.1

4 Balancedness of Highway Games on Weakly Cyclic Graphs

In this section, we determine sufficient conditions on a highway problem such that the induced highway game is balanced. We first focus on highway problems on single cycles and establish three sufficient conditions for the balancedness of the induced highway games.

Our first result states that if the number of players is less than or equal to three, then highway games on a cycle are balanced. Consider a highway problem $(N, G, \{\{s_i, t_i\}\}_{i \in N}, w)$, where G is a cyclic graph. Observe that each player has two alternative paths that connect his connection vertices in G. For player i, we denote an individually optimal path for i by P_i and the alternative path by Q_i. We first prove two preliminary results.

Lemma 4.1 *Let the highway problem* $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i \in N}, w)$ *be such that* $G = (V, E)$ *is a cyclic graph and let* $j, k \in N$*. Then,*

$$w(P_j \cap P_k) \leq w(E \backslash (P_j \cup P_k)).$$

Proof. P_j can be partitioned into two sets of edges: $P_j \cap P_k$ and $P_j \backslash P_k$. Similarly, Q_j can be partitioned into $Q_j \cap P_k = P_k \backslash P_j$ and $Q_j \backslash P_k = E \backslash (P_j \cup P_k)$: See also Figure 3.

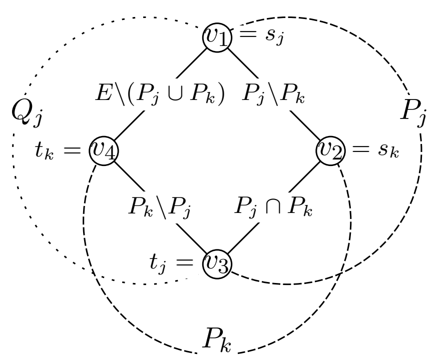

Figure 3: An auxiliary figure for the proof of Lemma 4.1

Since $w(P_j) \leq w(Q_j)$,

$$w(P_j) = w(P_j \cap P_k) + w(P_j \backslash P_k) \leq w(P_k \backslash P_j) + w(E \backslash (P_j \cup P_k)) = w(Q_j). \tag{2}$$

Similarly, since $w(P_k) \leq w(Q_k)$,

$$w(P_k) = w(P_j \cap P_k) + w(P_k \backslash P_j) \leq w(P_j \backslash P_k) + w(E \backslash (P_j \cup P_k)) = w(Q_k). \tag{3}$$

By summing inequalities (2) and (3), one obtains

$$w(P_j \cap P_k) \leq w(E \backslash (P_j \cup P_k)). \tag{4}$$

□

Lemma 4.2 *Let the highway problem* $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i \in N}, w)$ *be such that* $G = (V, E)$ *is a cyclic graph. Let* $j, k \in N$ $(j \neq k)$ *be such that* $w(P_j) \leq w(P_k)$*. Then,*

$$c_\Gamma(\{j, k\}) = \min\{w(P_j \cup P_k), w(P_j \cup Q_k)\}.$$

Proof. Clearly, $c_\Gamma(\{j, k\}) = \min\{w(P_j \cup P_k), w(P_j \cup Q_k), w(Q_j \cup P_k), w(Q_j \cup Q_k)\}$. We will show that $w(P_j \cup Q_k) \leq w(Q_j \cup P_k)$ and $w(P_j \cup P_k) \leq w(Q_j \cup Q_k)$. Note that $P_j \cup Q_k$ can be partitioned into two sets P_j and $Q_k \backslash P_j = E \backslash (P_j \cup P_k)$. Hence, $w(P_j \cup Q_k) = w(P_j) + w(E \backslash (P_j \cup P_k))$. Similarly, $w(Q_j \cup P_k) = w(P_k) + w(E \backslash (P_j \cup P_k))$. We know that $w(P_j) \leq w(P_k)$. Hence, $w(P_j \cup Q_k) \leq w(Q_j \cup P_k)$.

One can easily observe that $w(P_j \cup P_k) = w(E) - w(E \backslash (P_j \cup P_k))$ and $w(Q_j \cup Q_k) = w(E) - w(P_j \cap P_k)$. Then, by Lemma 4.1, $w(P_j \cup P_k) \leq w(Q_j \cup Q_k)$. □

Proposition 4.1 *Let* $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i \in N}, w)$ *with* $|N| \leq 3$ *be a highway problem such that* G *is a cyclic graph. Then the corresponding highway game* (N, c_Γ) *is balanced.*

Proof. If $|N| = 2$, balancedness of (N, c_Γ) follows from the subadditivity of the game. Set $N = \{1, 2, 3\}$. Without loss of generality, assume that $w(P_1) \leq w(P_2) \leq w(P_3)$. We will show that $x = (c_\Gamma(\{1\}), c_\Gamma(\{1, 2\}) - c_\Gamma(\{1\}), c_\Gamma(N) - c_\Gamma(\{1, 2\})) \in C(c_\Gamma)$. Since (N, c_Γ) is subadditive, we only need prove the core inequality corresponding to the coalition $\{1, 3\}$, i.e., we need to prove that

$$c_\Gamma(N) - c_\Gamma(\{1, 2\}) + c_\Gamma(\{1\}) \leq c_\Gamma(\{1, 3\}). \tag{5}$$

By Lemma 4.2, $c_\Gamma(\{1, 2\})$ is either equal to $w(P_1 \cup Q_2)$ or equal to $w(P_1 \cup P_2)$ and $c_\Gamma(\{1, 3\})$ is either equal to $w(P_1 \cup Q_3)$ or equal to $w(P_1 \cup P_3)$. Firstly, assume that $c_\Gamma(\{1, 2\}) = w(P_1 \cup P_2)$ and $c_\Gamma(\{1, 3\}) = w(P_1 \cup P_3)$. Then,

$$\begin{aligned} c_\Gamma(\{1, 2\}) + c_\Gamma(\{1, 3\}) - c_\Gamma(\{1\}) &= w(P_1 \cup P_2) + w(P_1 \cup P_3) - w(P_1) \\ &= w(P_1 \cup P_2) + w(P_3 \backslash P_1) \\ &\geq w(P_1 \cup P_2) + w(P_3 \backslash (P_1 \cup P_2)) \\ &= w(P_1 \cup P_2 \cup P_3) \geq c_\Gamma(N), \end{aligned} \tag{6}$$

where the second equality follows from the fact that $P_1 \cup P_3$ can be partitioned into sets P_1 and $P_3 \backslash P_1$. Analogously, the remaining cases regarding the choice of $c_\Gamma(\{1, 2\})$ and

$c_\Gamma(\{1,3\})$ lead to the same result. $\square$

In fact, in the proof of Proposition 4.1, we specify an allocation which always belongs to the core of a three person highway game on a cyclic graph. This allocation corresponds to a marginal of the highway game corresponding to an order in which the players are ordered on the basis of individual costs. The following example illustrates that, for highway games on cyclic graphs with more than three players, such a marginal need not be in the core.

Example 4.1 Consider the highway problem $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i \in N}, w)$ with $N = \{1, 2, 3, 4\}$ as depicted in Figure 4.

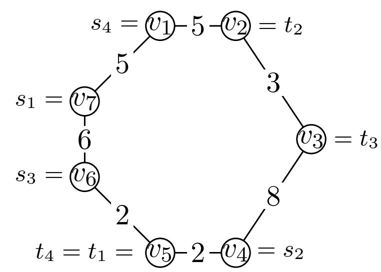

Figure 4: A highway problem with four players

The highway game (N, c_Γ) corresponding to this problem is given by:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
c_Γ(S)	8	11	12	13	19	18	13	15	18	20	21	20	21	20	23

Observe that $c_\Gamma(1) \le c_\Gamma(2) \le c_\Gamma(3) \le c_\Gamma(4)$. Then the marginal corresponding to the order (1234) in which players are ranked on the basis of their individual costs is $x = (8, 11, 2, 2)$. This marginal is not in $C(c_\Gamma)$, since $x_1 + x_2 + x_4 = 21 > 20 = c_\Gamma(\{1, 2, 4\})$. Note however that (N, c_Γ) is balanced. For example, the cost allocation $(6, 3, 12, 2)$ is in $C(c_\Gamma)$. $\diamond$

In the following proposition we show that if, in a highway problem defined on a cyclic graph, all players' individually optimal paths are disjoint, then the corresponding highway game is balanced.

Proposition 4.2 *Let $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i \in N}, w)$ be a highway problem such that G is a cyclic graph. If $P_i \cap P_j = \emptyset$ for all $i, j \in N$ with $i \neq j$, then the corresponding highway game (N, c_Γ) is balanced.*

Proof. Let $N = \{1, 2, ..., n\}$ and assume that $c_\Gamma(\{n\}) \ge c_\Gamma(\{i\})$ for every $i \in N$. It is easily observed that

$$c_\Gamma(S) = \begin{cases} w(\cup_{i \in S} P_i) & \text{if } n \in S \text{ and } w(P_n) \le w(E \setminus \cup_{i \in S} P_i) \\ w(Q_n) & \text{if } n \in S \text{ and } w(E \setminus \cup_{i \in S} P_i) \le w(P_n) \\ w(\cup_{i \in S} P_i) & \text{if } n \notin S. \end{cases}$$

Now, it is readily seen that the allocation in which every player $i \neq n$ pays $c_\Gamma(\{i\}) = w(P_i)$ and n pays $w(P_n)$ if $w(P_n) \leq w(E \backslash \cup_{i \in S} P_i)$ and $w(E \backslash \cup_{i \in S} P_i)$ if $w(E \backslash \cup_{i \in S} P_i) \leq w(P_n)$, is a core allocation of (N, c_Γ). $\square$

In the following proposition we will prove that for a highway problem defined on a cyclic graph, if each player wants to establish a connection with the same location, then the corresponding highway game is balanced.

Proposition 4.3 *Let $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i \in N}, w)$ be a highway problem such that G is a cyclic graph and $s_i = s_j$ for every $i, j \in N$. Then, the corresponding highway game (N, c_Γ) is balanced.*

Proof. Let $G = (V, E)$ be a cyclic graph with $V = \{v_0, v_1, ..., v_k\}$ $(k \geq 2)$ and $E = \{\{v_0, v_1\}, \{v_1, v_2\}, ..., \{v_k, v_0\}\}$. Assume that $s_i = s_j = v_0$ for every $i, j \in N$. Moreover, let $E^* \subset E$ be a set of edges that is optimal to construct for N, i.e., $c_\Gamma(N) = w(E^*)$. In the following, we will assign each player $i \in N$ the cost of a (possibly empty) subset E_i^* of E^* and show that the corresponding cost allocation is in the core of (N, c_Γ).

Firstly, let $N^v = \{i \in N | t_i = v\}$ for each $v \in V$. Now, consider a vertex $v_t \in V$ with $N^{v_t} \neq \emptyset$. Clearly, either the path $\{\{v_t, v_{t+1}\}, ..., \{v_k, v_0\}\}$ is in E^* or the path $\{\{v_t, v_{t-1}\}, ..., \{v_1, v_0\}\}$ is in E^* so that v_t and v_0 are connected in (V, E^*). Also E^* can always be decomposed into two (possibly empty) disjoint paths $\{\{v_0, v_1\}, \{v_1, v_2\} ..., \{v_{q-1}, v_q\}\}$ and $\{\{v_0, v_k\}, \{v_k, v_{k-1}\}, ..., \{v_{r+1}, v_r\}\}$ for some $v_q \in V$ and $v_r \in \{v_{q+1}, ..., v_0\}$.

We first consider the path $\{\{v_0, v_1\}, \{v_1, v_2\} ..., \{v_{q-1}, v_q\}\}$. Let v_{t_1} be the first vertex in the sequence $(v_0, v_1, ..., v_q)$ such that $N^{v_{t_1}} \neq \emptyset$, i.e., $N^{v_{t_1}} \neq \emptyset$ and $t_1 \leq t'$ for every $t' \in \{0, 1, ..., q\}$ such that $N^{v_{t'}} \neq \emptyset$. Pick a player $i \in N^{v_{t_1}}$ and set $E_i^* = \{\{v_0, v_1\}, ..., \{v_{t_1 - 1}, v_{t_1}\}\}$. Set $E_j^* = \emptyset$ for every $j \in N^{v_{t_1}} \backslash \{i\}$. If $t_1 = q$, then we are done. Otherwise, let v_{t_2} be the first vertex in the sequence $(v_{t_1+1}, ..., v_q)$ such that $N^{v_{t_2}} \neq \emptyset$. Pick a player $i \in N^{v_{t_2}}$ and set $E_i^* = \{\{v_{t_1}, v_{t_1+1}\}, ..., \{v_{t_2-1}, v_{t_2}\}\}$. Set $E_j^* = \emptyset$ for every $j \in N^{v_{t_2}} \backslash \{i\}$. If $t_2 = q$, then we are done. Otherwise, one can repeat the procedure above until v_q is reached. Notice that the sets of edges assigned to players form a partition of the path $\{\{v_0, v_1\}, \{v_1, v_2\} ..., \{v_{q-1}, v_q\}\}$. Obviously, one can apply a similar procedure to allocate the cost of the path $\{\{v_0, v_k\}, \{v_k, v_{k-1}\}, ..., \{v_{r+1}, v_r\}\}$.

We will now show that the cost allocation $x = (w(E_1^*), w(E_2^*), ..., w(E_n^*))$ is in the core of (N, c_Γ). One can easily observe that $\sum_{i \in N} w(E_i^*) = w(E^*) = c_\Gamma(N)$ by construction. Suppose that there exists a coalition $S \subset N$ such that $\sum_{i \in S} w(E_i^*) > c_\Gamma(S)$. Let $E_{N \backslash S}^* = \cup_{i \in N \backslash S} E_i^*$ and let E_S be a set of edges that is optimal to construct for S, i.e., $c_\Gamma(S) = w(E_S)$. Then,

$$c_\Gamma(N) = \sum_{i \in S} w(E_i^*) + \sum_{i \in N \backslash S} w(E_i^*) > c_\Gamma(S) + \sum_{i \in N \backslash S} w(E_i^*) = w(E_S) + w(E_{N \backslash S}^*). \quad (7)$$

By the construction of E^*, for each $i \in N \backslash S$, either there exists a path in $E_{N \backslash S}^*$ which connects t_i with v_0 or there exists a path in $E_{N \backslash S}^*$ which connects t_i with a vertex v such that $N^v \cap S \neq \emptyset$. Then, for every player $i \in N$, connection points s_i and t_i are connected via $(E^*_{N\setminus S} \cup E_S)$ and hence, (7) contradicts with the optimality of the edge set E^* for N. $\square$

We have obtained three sufficient conditions ensuring the balancedness of a highway game on a cyclic graph. In the following theorem, we show that, if each sub-highway problem with respect to a cycle (as defined in Section 3) in a weakly cyclic graph satisfies one of the three sufficiency conditions, then the highway game induced by the weakly cyclic graph is balanced.

Theorem 4.1 *Let $\Gamma = (N, G, \{\{s_i, t_i\}\}_{i\in N}, w)$ be a highway problem such that G is a weakly cyclic graph. The highway game (N, c_Γ) is balanced, if for every $C \in C(G)$, the sub-highway problem $\Gamma^C = (N, (V_C, C), \{\{s^C_i, t^C_i\}\}_{i\in N}, w_{|C})$ satisfies one of the following three conditions:*

(i) $|N| \leq 3$.

(ii) $P^C_i \cap P^C_j = \emptyset$ *for all* $i, j \in N$ *with* $i \neq j$.

(iii) $s^C_i = s^C_j$ *for every* $i, j \in N$.

Proof. We know by propositions 4.1, 4.2 and 4.3 that if a sub-highway problem induced by a cycle in G satisfies one of the conditions $(i), (ii)$ and (iii), then the corresponding sub-highway game is balanced. Moreover, by Lemma 3.2, a highway game on a weakly cyclic graph is equal to the sum of the sub-highway games with respect to each of its cycles and with respect to each bridge edge in the graph, i.e., $c_\Gamma(S) = \sum_{C\in C(G)} c_{\Gamma^C}(S) + \sum_{e\in BE(G)} c_{\Gamma^e}(S)$ for every $S \subset N$. Pick $y^C \in C(c_{\Gamma^C})$ for each $C \in C(G)$ and $y^e \in C(c_{\Gamma^e})$ for each $e \in BE(G)$. Set $x = \sum_{C\in C(G)} y^C + \sum_{e\in BE(G)} y^e$. Clearly, $x \in C(c_\Gamma)$. $\square$

5 Concluding Remarks

Most of the current literature on the allocation of the construction costs of networks focuses on minimum cost spanning tree (mcst) problems (cf. Granot and Huberman, 1981). These problems consider a group of players, each of whom has to be connected to a source, either directly or via other players. The main difference between highway problems and mcst problems is that, in a highway problem, there is no particular vertex every player has to be connected to. A difference less important is that usually, in mcst problems, a coalition S is not allowed to use vertices other than the vertices of S and the source. Indeed, if this restriction is relaxed, then the corresponding relaxed mcst game is a special type of highway game.

Moreover, for complete graphs, highway problems are related to minimum cost forest (mcf) problems introduced by Kuipers (1997). Mcf problems are generalizations of mcst problems which allow for more than one source, where each source offers a different type of service and each customer has to be connected with a nonempty subset of the available sources. Kuipers (1997) establishes sufficient conditions for the balancedness of the corresponding mcf games. Highway games on complete graphs form a subclass of the class of mcf games. As Kuipers (1997) provides an example of a three person mcf problem, in which every player has to be connected to exactly one source, that leads to a non-balanced game, it follows that a three person highway game on a complete graph need not be balanced.

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